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On continuous Moyal product structure in string field theory

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Abstract: We consider a diagonalization of Witten's star product for a ghost system of arbitrary background charge and Grassmann parity. To this end we use a bosonized formulation of such systems and a spectral analysis of Neumann matrices. We further identify a continuous Moyal product structure for a combined ghosts+matter system. The normalization of multiplication kernel is discussed.

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1. Introduction

A representation of Witten's star product in open string field theory (SFT) [1] as a family of infinitely many Moyal products was first proposed by I. Bars in [2]. The approach of that paper is essentially based on split formalism in SFT, it was further elaborated in papers [2, 3]. A different approach to Moyal-type representation for Witten's star product

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based on spectroscopy of Neumann matrices [4] was taken up in [5]. The authors of that paper showed that in the matter sector the Witten's star can be represented as a continuous Moyal product specified by a family of noncommutativity parameters

$$\theta(\kappa) = 2\tanh\left(\frac{\pi\kappa}{4}\right)$$

labelled by a continuous variable $0 \leq \kappa < \infty$. An additional assumption in that paper was the restriction to zero momentum sector. The whole construction was generalized to arbitrary momentum in [6].

One hopes that reformulation of string field theory that uses a continuous Moyal product will make the whole structure more transparent and introduce new computational tools. Given the fact that SFT axioms [1] mimic the conventional axioms of noncommutative geometry [7], the theory written in some new basis may ultimately resemble a lot noncommutative field theories (see [8]–[12] for a review). Thus in noncommutative field theories solitonic solutions were explicitly constructed [13] using Moyal algebra projectors. It turns out the well known sliver solution [14] in SFT being rewritten in the continuous Moyal basis is represented by a functional

$$\Xi[x_\alpha(\kappa)] = \mathcal{N}\exp\left[-\int_0^\infty d\kappa\,\frac{x_\alpha(\kappa)x_\alpha(\kappa)}{\theta(\kappa)}\right]$$

that has the form of a continuous family of noncommutative field theory projectors (see [15] for a detailed discussion of this representation of the sliver).

To complete the program and rewrite the whole SFT (VSFT or Witten's cubic SFT) in the continuous Moyal basis one needs to extend the formalism to include the ghost sector and a BRST operator. In the present paper we extend the approach of [5] to ghosts. A preliminary work in that direction was done in [6] where ghosts in the bosonic SFT were treated in the fermionic b-c-representation. In [16] a continuous Moyal product was constructed for string fields including ghosts (in bc formalism) restricted to Siegel's gauge. In this paper we work in the bosonized representation considering ghost systems of arbitrary background charge and Grassmann parity. We find working in bosonized representation technically simpler than treating bc- or $\beta\gamma$-systems directly. The continuous Moyal product for fermionic fields was considered in [17].

The paper is organized as follows. In section 2 we set up some notations and remind the reader the form of two and three-string vertices for ghosts in the bosonized representation. In section 3 we diagonalize these vertices using the spectroscopy of Neumannn matrices. In section 4 we represent the Witten's star product as a functional integral operator specified by a certain kernel. In section 5 we discuss regularization of determinants in the continuous Moyal basis. As an instructive exercise we check the cancellation of infinities for overlaps of surface states. In section 6 we apply the technique of section 5 to the normalization constant of the multiplication kernel. This section also contains further discussion of the precise correspondence between the Witten star product and the continuous Moyal one. We end with conclusions in section 7. Appendices contains technical details of the calculations.

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2. Ghost 2- and 3-string vertices in the bosonized formulation

2.1 Bosonized formulation

In bosonized formulation a ghost bc- or $\beta\gamma$- system is represented by a single bosonic field $\phi(z,\bar{z})$ with a mode expansion [18, 19]

$$\epsilon\phi(z,\overline{z}) = 2i\phi_0 + j_0 \log z\overline{z} - \sum_{n\neq 0} \frac{j_n}{n}\left(\frac{1}{z^n} + \frac{1}{\overline{z}^n}\right), \tag{2.1a}$$

where j_0 is the ghost number operator taking integral values $q \in \mathbb{Z}$, parameter ϵ takes values $+1$ for a fermionic b-c system and -1 for a bosonic β-γ ghost system. The parameter ϵ also enters the commutation relations

$$[\phi_0, j_0] = i\epsilon \qquad \text{and} \qquad [j_n, j_m] = \epsilon n \delta_{n+m,0}\,. \tag{2.1b}$$

A conformal stress-energy tensor for a field ϕ with a background charge Q is given by the expression

$$T_\phi = \frac{\epsilon}{2}\partial\phi\partial\phi - \frac{Q}{2}\partial^2\phi\,. \tag{2.1c}$$

The last term in the stress energy tensor is due to the anomaly in the conservation of the ghost current $j = \epsilon\partial\phi$. The number Q is a background charge, which is equal to -3 for conformal bc ghost system of the bosonic string.

Denote by $|q\rangle$ a state that is annihilated by j_n, $n > 0$ and is of ghost number q:

$$j_n|q\rangle = 0 \qquad \text{and} \qquad j_0|q\rangle = q|q\rangle\,. \tag{2.2}$$

Because of the anomaly in the conservation of the ghost charge the inner product is nonzero only for the following states

$$\langle -q - Q|q'\rangle = \delta_{qq'}\,. \tag{2.3}$$

We further introduce creation and annihilation operators according to

$$j_{-n} = \sqrt{n}\, a_n^\dagger \qquad \text{and} \qquad j_n = \sqrt{n}\, a_n\,, \quad n > 0\,. \tag{2.4a}$$

Notice that the commutation relations for these oscillators include ϵ

$$[a_n,\, a_m^\dagger] = \epsilon\delta_{nm}\,. \tag{2.4b}$$

2.2 Overlap vertices

The two operations: multiplication and inner product, needed to define string field theory action, can be written in terms of 3- and 2-string vertices. Here we will present only the vertices for the bosonized ghosts. The ones for the matter part can be found, for example in [20]–[22].

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The 3-string vertex for a bosonized field (2.1) can be written as [23, 24]

$$
\begin{aligned}
|V_3^{(\phi)}\rangle_{123} \;=\; & \sum_{q_r\in\mathbb{Z}} \delta_{q_1+q_2+q_3+Q,0}\;\times \\
& \times\exp\Bigg[-\frac{\epsilon}{4}V'_{00}\Big(\sum_r q_r^2 - Q^2\Big)+\frac{\epsilon}{\sqrt{2}}\sum_{r,s}\sum_{n=1}^{\infty} q_r V'^{rs}_{0n} a_n^{(s)\dagger} - \qquad (2.5a)\\
& -\frac{\epsilon}{2}\sum_{r,s}\sum_{n,m=1}^{\infty} a_n^{(r)\dagger} V'^{rs}_{nm} a_m^{(s)\dagger} - -\epsilon\frac{Q\sqrt{2}}{6}\sum_{r=1}^{3}\sum_{n=1}^{\infty} J_n a_n^{(r)\dagger}\Bigg]\bigotimes_{r=1}^{3}|-q_r-Q\rangle\,,
\end{aligned}
$$

where

$$V'^{rs}_{nm} \;=\; \frac{1}{3}\big(C' + \alpha^{s-r}U' + \alpha^{r-s}\bar{U}'\big)_{nm} \qquad (2.5b)$$

$$V'^{rs}_{0m} \;=\; \frac{1}{3}\big(\alpha^{s-r}W'_m + \alpha^{r-s}\bar{W}'_m\big) \qquad (2.5c)$$

$$V'_{00} \;=\; -2\log\gamma, \qquad \gamma = \frac{4}{3\sqrt{3}} \qquad (2.5d)$$

are the standard 3-string vertex matrices [20, 22], C'_{mn} is a twist matrix $(-1)^n\delta_{nm}$ and vector J_n is given by

$$J_{2n} = \frac{(-1)^n}{\sqrt{n}} \qquad \text{and} \qquad J_{2n+1} = 0\,. \qquad (2.5e)$$

The indices r and s in (2.5a) run from 1 to 3 and label the tensorial components in the 3-string Hilbert space. Essentially (2.5a) differs from the matter vertex by the presence of the background charge in the Kronecker symbol and the term linear in Q in the exponent which arises from the insertion of the operator [20]

$$\exp\left[-\frac{\epsilon Q}{12}\sum_{r=1}^{3}\phi^{(r)}\left(\frac{\pi}{2}\right)\right] \qquad (2.6)$$

into the vertex (2.5a) with $Q = 0$.[1]

As a side remark we would like to note that the exponent here is not normal ordered. Formally, if one acts by the operator (2.6) on the vertex (2.5a) with $Q = 0$, one obtains an extra numeric factor

$$\exp\left[-\frac{\epsilon Q^2}{24}\sum_{n=1}^{\infty} J_n^2\right].$$

This factor is important for the overall normalization of the vertex, our normalization coincides with the one in [23]. Notice also that the quadratic term in the vertex exponential is exactly the same as that of the matter part upon the substitution $\epsilon \mapsto g_{\mu\nu}$.

[1] The mid-point insertion for 3-string vertex in notations of [20] is equal to

$$\exp\left[-\frac{iQ}{2}\frac{1}{3}\sum_{r=1}^{3}\phi_{\rm GJ}^{(r)}\left(\frac{\pi}{2}\right)\right].$$

To obtain expression (2.6) we take into account that $\phi_{\rm GJ} = -\frac{i}{2}\phi_{\rm here}$.

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The 2-string vertex reads [23]

$$|V_2^{(\phi)}\rangle_{12} = \sum_{q\in\mathbb{Z}} \exp\left[-\epsilon \sum_{n=1}^{\infty} a_n^{(1)\dagger}(-1)^n a_n^{(2)\dagger}\right] |q\rangle_1 \otimes |-q-Q\rangle_2 \,. \tag{2.7}$$

It realizes the bpz conjugation on the modes of field ϕ.

3. Diagonalization

3.1 Notations

The diagonalization of the matter vertex (at zero momentum) found in [5] is based on the spectral analysis of Neumann matrices done in [4]. It was shown in [4] that the matrices $(C'V')^{rs}_{mn}$ have a joint set of eigenvectors $v_n^{(\kappa)}$ labeled by a continuous parameter $-\infty < \kappa < \infty$. Namely we have

$$\sum_n (C'V'^{rs})_{mn} v_n^{(\kappa)} = \mu^{rs}(\kappa) v_m^{(\kappa)} \,,$$

where

$$\mu^{rs}(\kappa) = \frac{1}{1+2\cosh(\pi\kappa)/2}\Big[1 - 2\delta_{r,s} + e^{(\pi\kappa)/2}\delta_{r+1,s} + e^{-(\pi\kappa)/2}\delta_{r,s+1}\Big] \,. \tag{3.1}$$

The vectors $v^{(\kappa)}$ are chosen to satisfy the following orthogonality and completeness relations [25]

$$(v^{(\kappa)}, v^{(\kappa')}) = \mathcal{N}(\kappa)\delta(\kappa-\kappa') \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{d\kappa}{\mathcal{N}(\kappa)} v_m^{(\kappa)} v_n^{(\kappa)} = \delta_{mn} \,, \tag{3.2}$$

where

$$\mathcal{N}(\kappa) = \frac{2}{\kappa}\sinh\left(\frac{\pi\kappa}{2}\right) . \tag{3.3}$$

We use $v^{(\kappa)}$ to introduce a new oscillator basis

$$a_\kappa^\dagger = \frac{1}{\sqrt{\mathcal{N}(\kappa)}} \sum_{n=1}^{\infty} a_n^\dagger v_n^{(\kappa)} \tag{3.4}$$

and the same for a_κ. The action of the operator C' and commutation relations are of the form

$$C' a_\kappa^\dagger = -a_{-\kappa}^\dagger \quad \text{and} \quad [a_\kappa,\, a_{\kappa'}^\dagger] = \epsilon\delta(\kappa-\kappa') \,. \tag{3.5}$$

We next perform one more change of the oscillator basis which diagonalizes the operator C'

$$\begin{aligned} e_\kappa^\dagger &= \frac{a_\kappa^\dagger + C' a_\kappa^\dagger}{\sqrt{2}} \,, \qquad o_\kappa^\dagger = \frac{a_\kappa^\dagger - C' a_\kappa^\dagger}{i\sqrt{2}} \quad \kappa \geqslant 0 \,; \\ a_\kappa^\dagger &= \frac{e_\kappa^\dagger + i o_\kappa^\dagger}{\sqrt{2}} \,, \qquad C' a_\kappa^\dagger = \frac{e_\kappa^\dagger - i o_\kappa^\dagger}{\sqrt{2}} \quad \kappa \geqslant 0 \,. \end{aligned} \tag{3.6a}$$

In addition one has

$$o_{-\kappa}^\dagger = o_\kappa^\dagger \qquad \text{and} \qquad e_{-\kappa}^\dagger = -e_\kappa^\dagger \,. \tag{3.6b}$$

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Notice that for $\kappa = 0$ there is only one non-vanishing mode $o^\dagger_{\kappa=0}$. The bpz conjugation specified by (2.7) acts on these oscillators as

$$\mathrm{bpz}\, e_\kappa = -e^\dagger_\kappa \qquad \text{and} \qquad \mathrm{bpz}\, o_\kappa = -o^\dagger_\kappa \tag{3.7}$$

and the commutation relations are

$$[e_\kappa,\, e^\dagger_{\kappa'}] = \epsilon\delta(\kappa-\kappa') \qquad \text{and} \qquad [o_\kappa,\, o^\dagger_{\kappa'}] = \epsilon\delta(\kappa-\kappa')\,. \tag{3.8}$$

3.2 3-string and 2-string vertices

A straightforward computation yields the following form of the 3-string vertex in the new basis (see appendix B for details)

$$\begin{aligned}|V_3^{(\phi)}\rangle_{123} \,=\, &\sum_{q_r}\delta_{q_1+q_2+q_3+Q,0}\,\times\\ &\times\exp\left[\int_0^\infty d\kappa\left(-\frac{\epsilon}{4}J_\kappa^2(1+3\mu)\left[\sum_r q_r^2-Q^2\right]-\epsilon\sum_r\Phi^{(r)}_\alpha(\kappa)a^{(r)\dagger}_{\kappa,\alpha}-\right.\right.\\ &\left.\left.-\frac{\epsilon}{2}a^{(r)\dagger}_{\kappa,\alpha}V^{rs}_{\kappa,\alpha\beta}a^{(s)\dagger}_{\kappa,\beta}\right)\right]\bigotimes_{r=1}^{3}|-q_r-Q\rangle\,,\end{aligned} \tag{3.9}$$

where the quadratic part in the exponent is given by an operator

$$V^{rs}_{\kappa,\alpha\beta} \,=\, \mu\delta_{\alpha\beta}\otimes\delta^{rs}+\mu_s\delta_{\alpha\beta}\otimes\varepsilon^{rs}+i\mu_a\epsilon_{\alpha\beta}\otimes\chi^{rs}\,, \tag{3.10a}$$

$$\mu_s \,=\, \frac{1}{2}(\mu^{12}+\mu^{21})\,, \qquad \mu_a = \frac{1}{2}(\mu^{12}-\mu^{21})\,, \qquad \mu\equiv\mu^{11} \tag{3.10b}$$

and we introduced the following matrices

$$\varepsilon^{rs} = \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}, \qquad \chi^{rs} = \begin{pmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{pmatrix}, \qquad \epsilon_{\alpha\beta} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{3.10c}$$

The matrices $1, \varepsilon$ and χ form a closed algebra

$$\chi^2 = \varepsilon - 2\,, \qquad \varepsilon^2 = \varepsilon + 2\,, \qquad \chi\varepsilon = \varepsilon\chi = -\chi\,. \tag{3.10d}$$

The indices α, β in the above expressions take values "e" and "o" for even and odd modes (3.6a) respectively. Thus for instance $a^{(r)\dagger}_{\kappa,o} \equiv o^{(r)\dagger}_\kappa$ and $a^{(r)\dagger}_{\kappa,e} \equiv e^{(r)\dagger}_\kappa$. Since the transformation $a_n \to a_{\kappa,\alpha}$ is an orthogonal one, the vacuum state $|-q-Q\rangle$ in (3.9) stands for the tensor product of $a^{(r)}_{\kappa,\alpha}$-oscillators vacua and the state with the ghost number $-q-Q$ from the Hilbert space corresponding to the operators ϕ_0, j_0. The constant terms in (3.9) and the terms linear in the creation operators are expressed via

$$\Phi^{(r)}_e(\kappa) \,=\, \frac{1}{2}J_\kappa(1+3\mu)\left(q^{(r)}+\frac{Q}{3}\right)+\frac{1}{3}J_\kappa Q \tag{3.10e}$$

$$\Phi^{(r)}_o(\kappa) \,=\, iJ_\kappa\mu_a q^{(s)}\chi^{sr} \equiv \frac{2i}{\theta(\kappa)}\,\Phi^{(s)}_e\chi^{sr}\,, \tag{3.10f}$$

$$J_\kappa \,=\, \frac{1}{\sqrt{\mathcal{N}(\kappa)}}\,(v^{(\kappa)},J) = \frac{\sqrt{2}}{\kappa\sqrt{\mathcal{N}(\kappa)}}\,. \tag{3.10g}$$

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4. Multiplication Kernel

4.1 Coordinate basis

We now want to go to the coordinate representation corresponding to modes e_κ, o_κ. The coordinate eigenstates are

$$\langle X, q| = \langle q| \exp\left[\int_0^\infty d\kappa \sum_\alpha \left(-\frac{\epsilon}{2}\, x_\alpha(\kappa) x_\alpha(\kappa) + i\epsilon\sqrt{2}\, a_{\kappa,\alpha} x_\alpha(\kappa) + \frac{\epsilon}{2}\, a_{\kappa,\alpha} a_{\kappa,\alpha}\right)\right], \quad (4.1a)$$

where $\langle q|$ stand for the tensor product of $a^{(r)}_{\kappa,\alpha}$-oscillators bra-vector vacua with ghost number q. The bpz-conjugated state is then given by the expression

$$|X, q\rangle = \exp\left[\int_0^\infty d\kappa \sum_\alpha \left(-\frac{\epsilon}{2} x_\alpha(\kappa) x_\alpha(\kappa) - i\epsilon\sqrt{2} a^\dagger_{\kappa,\alpha} x_\alpha(\kappa) + \frac{\epsilon}{2} a^\dagger_{\kappa,\alpha} a^\dagger_{\kappa,\alpha}\right)\right] |q\rangle\,. \quad (4.1b)$$

One can rewrite the coordinate state (4.1b) in terms of the original discrete oscillator basis $a^\dagger_n$. To this end we introduce a vector

$$X_n = \int_{-\infty}^\infty \frac{d\kappa}{\mathcal{N}(\kappa)}\, v_n^{(\kappa)} X_\kappa\,, \quad (4.2a)$$

where

$$X_\kappa = \frac{x_e(\kappa) + i x_o(\kappa)}{\sqrt{2}} \qquad \text{and} \qquad -X_{-\kappa} = \frac{x_e(\kappa) - i x_o(\kappa)}{\sqrt{2}}\,. \quad (4.2b)$$

To understand the meaning of the vector X_n it is useful to compare it with notations of [5]. Using the fact that $x_e(\kappa)$ here is equal to $x(\kappa)$ in [5] and the same for $x_o(\kappa)$ here and $y(\kappa)$ there, one can obtain the following relations

$$X_{2n} = \sqrt{2n}\, x_{2n} \qquad \text{and} \qquad X_{2n-1} = -\frac{i}{\sqrt{2n-1}}\, p_{2n-1}\,, \quad (4.3)$$

where x_n and p_n are Fourier modes for the field $\phi(\sigma)$ and its momentum $P(\sigma)$

$$a\epsilon\phi(\sigma) = 2i\phi_0 + \sqrt{2}\sum_{n=1}^\infty x_n \cos n\sigma \qquad \text{and} \qquad \epsilon\pi P(\sigma) = j_0 + \sqrt{2}\sum_{n=1}^\infty p_n \cos n\sigma\,.$$

One can check then that the state (4.1b) takes the following form in terms of oscillators $a^\dagger_n$ and the coordinates x_{2n} and p_{2n-1}

$$\begin{aligned} |X, q\rangle = \exp&\left[\sum_{n=1}^\infty \left(-\frac{\epsilon}{2}(2n) x_{2n}^2 - i\epsilon\sqrt{2}\, a^\dagger_{2n}\sqrt{2n}\, x_{2n} + \frac{\epsilon}{2} a^\dagger_{2n} a^\dagger_{2n}\right)\right] \times \\ &\times \exp\left[\sum_{n=1}^\infty \left(-\frac{\epsilon}{2}\frac{1}{2n-1} p_{2n-1}^2 + \frac{\epsilon\sqrt{2}}{\sqrt{2n-1}}\, a^\dagger_{2n-1} p_{2n-1} - \frac{\epsilon}{2} a^\dagger_{2n-1} a^\dagger_{2n-1}\right)\right] |q\rangle\,. \end{aligned} \quad (4.4)$$

The calculations of the inner product of two such states yields

$$\langle X, -q-Q | X', q\rangle = \prod_{n=1}^\infty \left[\frac{\pi}{2n\epsilon}\right]^{1/2} \delta(x_{2n} - x'_{2n}) \times \prod_{n=1}^\infty \left[\frac{\pi(2n-1)}{\epsilon}\right]^{1/2} \delta(p_{2n-1} - p'_{2n-1})\,.$$

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Using this expression we can define the measure $\mathcal{D}X$ as

$$\mathcal{D}X = \prod_{n=1}^{\infty} \frac{dx_{2n}\, dp_{2n-1}}{\epsilon\pi} \left[\frac{2n-1}{2n}\right]^{-1/2} . \tag{4.5}$$

This measure coincides with the one used in [5].

With these notations we have the following representation for a unit operator in the ghost number q component of the Hilbert space $\mathcal{H}_q$

$$\mathrm{Id}_q = \int \mathcal{D}X\, |X, q\rangle \otimes \langle X, -q-Q| . \tag{4.6}$$

4.2 String product in the coordinate basis

Using representation (4.6) for the identity operator we can rewrite the string multiplication in the coordinate basis. In this representation the product is realized as a product of wave functionals $\Psi_q[X]$, which are related to the states from $\mathcal{H}_q$ via

$$\Psi_q[X] = \langle \Psi_q | X, -q-Q\rangle . \tag{4.7}$$

Then the multiplication of two functionals Ψ_{q_1} and Ψ_{q_2} with ghost numbers q_1 and q_2 respectively gives a functional $\Psi_{q_1} \star \Psi_{q_2}$ of ghost number q_1+q_2 given by

$$(\Psi_{q_1} \star \Psi_{q_2})[X^{(3)}] = \int \mathcal{D}X^{(1)} \mathcal{D}X^{(2)}\, K_{q_1,q_2}(X^{(1)}, X^{(2)}, X^{(3)}) \Psi_{q_1}[X^{(1)}] \Psi_{q_2}[X^{(2)}] . \tag{4.8}$$

Here the multiplication kernel K_{q_1,q_2} is given by the appropriate overlaps of the 3-string vertex with the coordinate eigenstates (see appendix C for details)

$$\begin{aligned} K_{\{q_1,q_2\}}(X^{(r)}) &\equiv \Big(\langle X^{(1)}, q_1| \otimes \langle X^{(2)}, q_2| \otimes \langle X^{(3)}, -Q-q_1-q_2|\Big)|V_3\rangle \\ &= \mathcal{K}_Q \exp\left[2i\epsilon \int_0^\infty \frac{d\kappa}{\theta(\kappa)}\, x_e^{(r)}(\kappa) \chi^{rs} x_o^{(s)}(\kappa)\right] \times \\ &\quad \times \exp\left[-i\epsilon \int_0^\infty d\kappa\, \sqrt{2} J_\kappa \left\{ x_e^{(1)}(\kappa)\Big(q_1 + \frac{Q}{2}\Big) + x_e^{(2)}(\kappa)\Big(q_2 + \frac{Q}{2}\Big) - \right.\right. \\ &\qquad \left.\left. - x_e^{(3)}(\kappa)\Big(\bar{q}_3 + \frac{Q}{2}\Big)\right\}\right] , \end{aligned} \tag{4.9a}$$

where $\bar{q}_3 = q_1 + q_2$, the non-commutativity parameter $\theta(\kappa)$ is

$$\theta(\kappa) = 2 \tanh\left(\frac{\pi\kappa}{4}\right) \tag{4.9b}$$

and $\mathcal{K}_Q$ is a normalization constant that can be formally written as

$$\mathcal{K}_Q = \frac{1}{\det 2(1+3\mu(\kappa))} \exp\left[\frac{\epsilon Q^2}{24} \sum_{n=1}^{\infty} J_n^2 - \frac{\epsilon Q^2}{3} \log\gamma\right] . \tag{4.9c}$$

We will discuss this constant in greater detail in the forthcoming sections.

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Let us notice here that the multiplication kernel for the bosonized ghosts differs from the one for the matter part (with $D = 1$) only by the presence of symbol Q in the exponent. Essentially if one considers the kernel for the matter sector with *non-zero* momentum one will obtain formula (4.9a) with $Q = 0$. For future use we remind the expression for zero momentum kernel in the matter sector

$$K^{\text{matter}}(X^{(r)}) = \mathcal{K}^D_{Q=0} \exp\left[\int_0^\infty d\kappa\, \frac{2i}{\theta(\kappa)} \sum_{\mu=0}^{D-1} x^{(r)}_{\mu,e} \chi^{rs} x^{(s)}_{\mu,o}\right]. \tag{4.10}$$

It is clear from the $+, +, -$ structure of the additional exponential term in the ghost kernel (4.9a) that it still defines an associative multiplication in the ghost sector and moreover this additional factor can be removed by a wave-function redefinition:

$$\tilde{\Psi}_q[X] = \exp\left[-i\sqrt{2}\epsilon\Big(q + \frac{Q}{2}\Big)\int_0^\infty d\kappa\, J_\kappa x_e(\kappa)\right] \Psi_q[X]\,. \tag{4.11}$$

The wave functions $\tilde{\Psi}_q(X)$ are multiplied with the help of the kernel (4.10) with $D = 27$, where $\mu = 26$ stands for the bosonized ghosts fields, and the normalization constant should be $\mathcal{K}_{Q=-3}\mathcal{K}^{26}_{Q=0}$.

The rest of the paper is essentially devoted to the study of the normalization constant (4.9c). The total constant $\mathcal{K}_{Q=-3}\mathcal{K}^{26}_{Q=0}$ contains two formally divergent factors. One comes from a determinant of a certain operator and the other one, proportional to the background charge Q^2 originates from the ghost anomaly insertion in the vertex. It was suggested in [5] that in the critical bosonic string theory, i.e. when $D = 26$, $Q = -3$, $\epsilon = 1$ and the combined central charge vanishes, the two terms may cancel each other.

Indeed cancellations of this form occur in SFT [23, 26]. In the next section we consider in detail one example of such cancellation that has to do with overlaps of surface states. We will discuss a general method for regularization of infinite determinants of operators written in the continuous basis $v_n^{(\kappa)}$ and check the cancellation noted in [23].

5. Overlaps of surface states

5.1 Surface state $|0\rangle \star |0\rangle$

One can consider a so called surface state [23, 27, 28] that has a form

$$\langle f, -q - Q| \equiv \langle -q - Q|U_f\,, \qquad U_f = \exp\left(\sum_{n=1}^\infty v_n L_n\right), \tag{5.1}$$

where v_n are specify a vector field $v(z) = \sum_n v_n z^n$ specifying a finite conformal transformation

$$f(z) = e^{v(z)\partial_z} z$$

(we also assumed a particular choice of the $SL(2,\mathbb{R})$-frame for which $f(0) = 0$).

The bpz-conjugated state is

$$|I \circ f \circ I, -q - Q\rangle \equiv U^{-1}_{I\circ f\circ I}|-q-Q\rangle\,, \qquad U^{-1}_{I\circ f\circ I} = \exp\left(\sum_{n=1}^\infty (-1)^n v_n L_{-n}\right),$$

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where I stands for the inversion mapping: $I(z) = -1/z$. The claim is that the scalar product of any two surface states in the CFT with zero central charge is equal to 1

$$\langle f, -Q | I \circ g \circ I, 0 \rangle = 1 \,. \tag{5.2}$$

There is a well known perturbative reason for this [23]. If we expand in series the operator U defined by (5.1) then the scalar product (5.2) will be given by a sum of the correlation functions involving only the stress energy tensor. In a theory with vanishing central charge all such correlators vanish, and therefore we have only one non-zero term in the sum, which is equal to 1.

Let us discuss in detail how we can check (non-perturbatively) (5.2) for the case of a special surface state $|0\rangle \star |0\rangle$, which corresponds to the map [27]

$$f(z) = f_1^{(3)} \equiv \left(\frac{1-iz}{1+iz}\right)^{2/3} .$$

Written in the oscillator representation the ghost parts of the corresponding surface states read

$$\langle f_1^{(3)}, -q-Q| = \langle -q-Q| \exp\left[-\frac{\epsilon}{2}\sum_{n,m=1}^{\infty} a_n V'^{11}_{nm} a_m - \frac{\epsilon q}{\sqrt{2}}\sum_{n=1}^{\infty} V'^{11}_{0n} a_n + \frac{\epsilon Q}{3\sqrt{2}}\sum_{n=1}^{\infty} J_n a_n\right], \tag{5.3a}$$

where for generality we wrote the state for an arbitrary vacuum ghost number [23]. Strictly speaking such a state with $q \neq 0$ should not be called a surface state as it is not built over the conformal vacuum. However insertions proportional to the background charge are always needed to get a non-vanishing overlap between two surface states. The terms proportional to Q that occur in (5.3a), (5.3b) serve precisely that purpose. (So we might as well call these states "modified surface states", or "surface states with insertions".)

The state bpz-conjugated to (5.3a) is of the form

$$|I \circ f_1^{(3)} \circ I, -q-Q\rangle = \exp\left[-\frac{\epsilon}{2}\sum_{n,m=1}^{\infty} a_n^\dagger V'^{11}_{nm} a_m^\dagger + \frac{\epsilon q}{\sqrt{2}}\sum_{n=1}^{\infty} V'^{11}_{0n} a_n^\dagger - \frac{\epsilon Q}{3\sqrt{2}}\sum_{n=1}^{\infty} J_n a_n^\dagger\right] \times \\ \times |-q-Q\rangle \,. \tag{5.3b}$$

Now using the functional integral technique [33] we find that the scalar product (5.2) is

$$\langle f_1^{(3)}, -Q | I \circ f_1^{(3)} \circ I, 0\rangle = \det\left[1-(M')^2\right]^{-\frac{D+1}{2}} \times \\ \times \exp\left[\frac{\epsilon Q^2}{4}\left(-\frac{2}{9} J_n \left([1-M']^{-1}\right)_{nm} J_m - \frac{2}{3} M'_{0n}\left([1-M']^{-1}\right)_{nm} \times \right.\right. \\ \left.\left. \times J_m - M'_{0n}\left([1-(M')^2]^{-1}M'\right)_{nm} M'_{0m}\right)\right], \tag{5.4}$$

where $M'_{nm} = (-1)^n V'^{11}_{nm}$ and $M'_{0n} = (-1)^n V'^{11}_{0n}$. One can further simplify the above expressions using the identities

$$\frac{1}{1-M'} = \frac{1+3M'}{4(1-M')} + \frac{3}{4} \qquad \text{and} \qquad M'_{0n} = -\frac{1}{3}\left(1+3M'\right)_{nm} J_m$$

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(see appendix A). Finally we get

$$\langle f_1^{(3)}, -Q|I\circ f_1^{(3)}\circ I, 0\rangle = \det\left[1-(M')^2\right]^{-\frac{D+1}{2}} \times \\ \times \exp\left[\frac{\epsilon Q^2}{4}\left(-\frac{1}{6}\sum_{n=1}^{\infty} J_n^2 - \frac{1}{2}M'_{0n}\left([1-M']^{-1}\right)_{nm} J_m - \right.\right. \\ \left.\left. - \sum_{n,m} M'_{0n}\left([1-(M')^2]^{-1}M'\right)_{nm} M'_{0m}\right)\right]. \quad (5.5)$$

Here we rearranged the terms in the exponent so that the divergent part $\sum_{n=1}^{\infty} J_n^2$ is singled out. The rest of the terms in the exponent are finite. This will be clear once we will rewrite these terms in the $v_n^{(\kappa)}$ basis in section 5.3. In that basis these terms are represented by convergent integrals.

Our goal now is to check that expression (5.5) is equal to 1 in the case of critical bosonic string, i.e. $\epsilon = 1$ $Q = -3$ and $D = 26$. We would like to note that this identity is a particular case of formula [23, (5.60)].

5.2 Regularization of determinants. Cancellation of infinities

Consider a symmetric operator G_{nm} acting in space of infinite sequences l_∞ labelled by the string mode index $n = 1, 2, \ldots$. Assume further that this operator takes a diagonal form in the basis $v_n^{(\kappa)}$ and its eigenvalues are $G(\kappa)$. Then its determinant can be formally written as

$$\det G_{nm} = \exp\bigl(\mathrm{Tr}\log G_{nm}\bigr) = \exp\left[\int_{-\infty}^{\infty} d\kappa\,\rho(\kappa)\log G(\kappa)\right], \quad (5.6a)$$

where $\rho(\kappa)$ is a spectral density. If the operator G_{nm} is rewritten in the basis of even and odd eigenvectors, its determinant will have the following form[2]

$$\det G_{\alpha\beta}(\kappa) = \exp\bigl(\mathrm{Tr}\log G_{\alpha\beta}(\kappa)\bigr) = \exp\left[\int_{0}^{\infty} d\kappa\,\rho(\kappa)\log G(\kappa)\right] \quad (5.6b)$$

where $\rho(\kappa)$ is the same spectral density as in (5.6a) and the indices α, β take values "o", "e".

It was suggested in [4, 25] that the regularized spectral density is $\rho_L(\kappa) = \log L/(2\pi)$ where L is a "level regulator" that truncates only the mode labels $m, n \leqslant L$. Strictly speaking this defines only the divergent part of the spectral density. The whole spectral density regulated by L can be written as

$$\rho_{2L}(\kappa) = \frac{1}{\mathcal{N}(\kappa)}\sum_{n=1}^{2L} v_n^{(\kappa)} v_n^{(\kappa)} = \frac{1}{2\pi}\sum_{n=1}^{L}\frac{1}{n} + \rho_{\mathrm{fin}}^{2L}(\kappa)\,. \quad (5.7)$$

The first term in this expression gives (up to a finite constant) the spectral density $\log L/(2\pi)$ used in [4, 25]. The second term is finite in the limit $L \to \infty$. We will discuss it in more detail in the next subsection concentrating for now on the divergent part.

[2] Notice that if $v^{(\kappa)}$ are eigenvectors of a symmetric operator G_{nm} then $G_{\alpha\beta}(\kappa) = G(\kappa)\delta_{\alpha\beta}$ and $G(-\kappa) = G(\kappa)$.

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L	32	52	62	82	∞
$\int$	-0.03550	-0.03558	-0.03561	-0.03563	-0.03602

Table 1: Here we present the numeric values of the quickly convergent integral $\int_{-\infty}^{\infty} d\kappa\, \rho_{\rm fin}^{2L}(\kappa) \log\bigl(1-\mu^2\bigr)$ computed at level L. In the very right column we present the value for this integral estimated by the formula $a + b/\log L$.

Using (5.6) and (5.7) we can write the following expression for the determinant appearing in (5.5)

$$\begin{aligned}\det\left[1-(M')^2\right] &= \exp\left[\int_{-\infty}^{\infty} d\kappa\, \rho(\kappa) \log\bigl(1-\mu^2\bigr)\right] \\ &= \lim_{L\to\infty} \exp\left[-\frac{1}{36}\sum_{n=1}^{L}\frac{1}{n} + \int_{-\infty}^{\infty} d\kappa\, \rho_{\rm fin}^{2L}(\kappa) \log\bigl(1-\mu^2\bigr)\right]. \end{aligned} \tag{5.8}$$

To obtain this equation we have used

$$\int_{-\infty}^{\infty} d\kappa\, \log\bigl(1-\mu^2\bigr) = -\frac{\pi}{18}\,.$$

The divergent part in the exponential term standing in (5.5) is

$$\exp\left[-\frac{\epsilon Q^2}{24}\sum_{n=1}^{\infty} J_n^2\right] \sim \exp\left[-\frac{\epsilon Q^2}{24}\sum_{n=1}^{L}\frac{1}{n}\right]. \tag{5.9}$$

Collecting both divergent terms from (5.8), (5.9) we obtain for the divergent part of the expression (5.5)

$$\exp\left[\left(\frac{D+1}{72} - \frac{\epsilon Q^2}{24}\right)\sum_{n=1}^{L}\frac{1}{n}\right]$$

that indeed vanishes when we plug in the parameters corresponding to the critical bosonic string theory ($D = 26$, $\epsilon = 1$ and $Q = -3$).

5.3 Finite parts

To compute the finite part of the determinant (5.8) we need to know the finite part $\rho_{\rm fin}(\kappa)$ of the spectral density. So far we could not find a closed analytic expression for $\rho_{\rm fin}(\kappa)$ and will present here only some numeric results. Figure 1 gives a plot of $\rho_{\rm fin}^{2L}(\kappa)$ in the vicinity of 0 computed for $L = 62$.

The integral in the r.h.s. of (5.8) converges rather quickly and can be easily computed numerically. We compute this integral on the interval $(-30, 30)$ for several values of L (see table 1). All this values are in a good agrement this the interpolating formula $a + b/\log L$.

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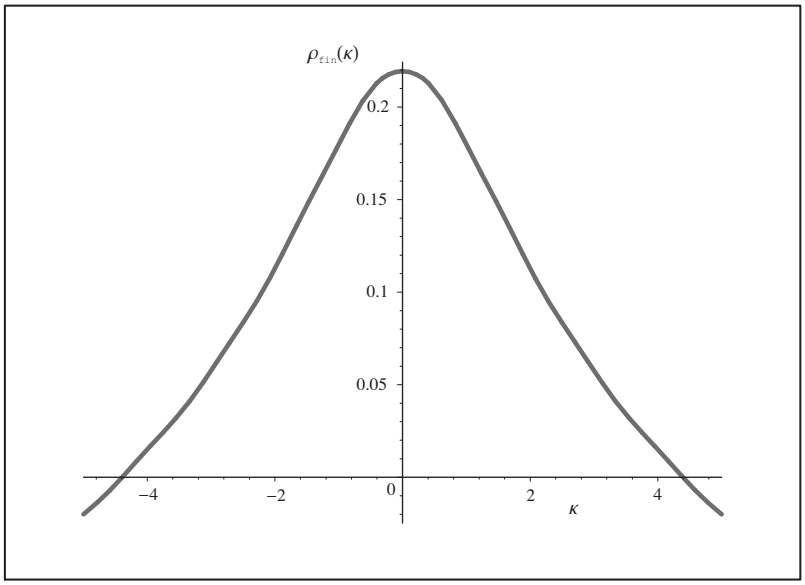

Figure 1: Plot of the finite part of the spectral density $\rho^{2L}_{\rm fin}(\kappa)$ computed at the level $L = 62$. $\rho_{\rm fin}(0) = \log 2/\pi$.

The finite part of the exponent in (5.5) can be computed by summing up residues in the upper half plane

$$\begin{aligned}(\text{finite}) &= \frac{\epsilon Q^2}{4}\left(-\frac{1}{2}M'_{0n}\left([1-M']^{-1}\right)_{nm}J_m - \sum_{n,m} M'_{0n}\left([1-(M')^2]^{-1}M'\right)_{nm}M'_{0m}\right)\\ &= \frac{\epsilon Q^2}{12}\int_{-\infty}^{\infty}\frac{d\kappa}{\mathcal{N}(\kappa)}\frac{2}{\kappa^2}\frac{1+3\mu}{1-\mu}\left[\frac{1}{2}-\frac{1}{3}\frac{1+3\mu}{1+\mu}\mu\right]\\ &= -\frac{\epsilon Q^2}{3}\log\gamma - \frac{\epsilon 7 Q^2}{72\pi^2}\zeta(3). \end{aligned} \tag{5.10}$$

If we substitute all of our results (5.8), (5.10) into equation (5.5) we obtain

$$\begin{aligned}\log\langle f_1^{(3)}, -Q|I\circ f_1^{(3)}\circ I, 0\rangle = \lim_{L\to\infty}&\left(\frac{D+1}{72}-\frac{\epsilon Q^2}{24}\right)\sum_{n=1}^{L}\frac{1}{n} -\\ &-\frac{D+1}{2}\int_{-\infty}^{\infty}d\kappa\,\rho^{2L}_{\rm fin}(\kappa)\log(1-\mu^2) -\\ &-\frac{\epsilon Q^2}{3}\log\gamma - \frac{\epsilon 7 Q^2}{72\pi^2}\zeta(3)\,,\end{aligned}$$

where for completeness we wrote both divergent and finite parts. As already noted the divergent parts cancel out for $Q = -3$, $D = 26$, $\epsilon = 1$. The value of the finite part can be estimated using the results presented in table 1. It turns out to be close to the number 1.1605 that seams to be far from the wanted 0. We hope that the origin of this difference is just an artifact of our numeric calculations. An analytic expression for the finite part of the spectral density would certainly come handy in proving the exact identity (5.2).

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6. Continuous Moyal versus Witten's star product

6.1 Normalization of the multiplication kernel

The total multiplication kernel combining the matter and ghost sectors contains an overall normalization constant

$$\mathcal{K}_{Q=0}^{D}\mathcal{K}_{Q} = \det\left[\frac{2}{\theta^2(\kappa)}\right]^{D+1} \times \det\left[\frac{12+\theta^2(\kappa)}{16}\right]^{D+1} \exp\left[\epsilon\frac{Q^2}{24}\sum_{n=1}^{\infty} J_n^2 - \epsilon\frac{Q^2}{3}\log\gamma\right]. \quad (6.1)$$

We can now apply the regularization technique of the previous section to compute this constant. We will keep the first factor untouched for the reason to be discussed a little later. The divergent part contained in the second two factors reads

$$\exp\left[\left(\frac{D+1}{2\pi}\int_0^{\infty} d\kappa\, \log\left(\frac{12+\theta^2(\kappa)}{16}\right) + \frac{\epsilon Q^2}{24}\right)\sum_{n=1}^{L}\frac{1}{n}\right] = \exp\left[\left(-\frac{D+1}{18} + \frac{\epsilon Q^2}{24}\right)\sum_{n=1}^{L}\frac{1}{n}\right] \quad (6.2)$$

and we see that for the parameters corresponding to the critical bosonic string these infinities do not cancel each other. This agrees with an analogous computation made in [16]. In this calculations we use

$$\int_0^{\infty} d\kappa\, \log\left(\frac{12+\theta^2(\kappa)}{16}\right) = -\frac{\pi}{9}.$$

We must note here that there is a potential subtlety in the above computation having to do with the finite part of the spectral density. It contributes the exponent

$$\exp\left[(D+1)\int_0^{\infty} d\kappa\, \rho_{\text{fin}}(\kappa)\log\left(\frac{16}{12+\theta^2(\kappa)}\right)\right]$$

which we attributed to the finite part. Strictly speaking we do not know the asymptotics of the function $\rho_{\text{fin}}(\kappa)$ as κ goes to infinity. However given the fact that it is being integrated with a factor that is exponentially falling off at infinity it looks unlikely that the integral diverges. We hope to clarify this subtlety in a future work.

6.2 Relation between continuous Moyal and string products

In this section we will try to state clearly the precise correspondence between Witten's and continuous Moyal star products. We start by reminding that the Moyal product of two functions on $\mathbb{R}^2$ can be defined by a kernel $K_\theta(x_1, x_2, x_3)$ in the following way

$$(f \star g)(x_1) = \int_{\mathbb{R}^2} dx_2 dx_3\, K_\theta(x_1, x_2, x_3) f(x_2) g(x_3)\,, \quad (6.3a)$$

where

$$K_\theta(x_1, x_2, x_3) = \frac{1}{\pi^2\theta^2}\exp\left[-\frac{2i}{\theta}\epsilon_{\alpha\beta}\left(x_1^\alpha x_2^\beta + x_2^\alpha x_3^\beta + x_3^\alpha x_1^\beta\right)\right] \quad (6.3b)$$

θ is a real deformation parameter and matrix $\epsilon_{\alpha\beta}$ is defined in (3.10c). Notice that one needs the factor θ^{-2} in the kernel to obtain a kernel for pointwise multiplication in the limit $\theta \to 0$.

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The continuous Moyal product will be defined as a product of functionals $\mathcal{F}[X^{\alpha,\mu}(\kappa)]$, where the functions $X^{\alpha,\mu}(\kappa) : \mathbb{R}_+ \to \mathbb{R}^2 \otimes \mathbb{R}^D$ ($\alpha = e, o$ and $\mu = 0, \ldots, D-1$) will be considered as canonical coordinates on our non-commutative space. Then the product can be defined as follows

$$(\mathcal{F}_1 * \mathcal{F}_2)[X^{(3)}] = \int \mathcal{D}X^{(1)}\mathcal{D}X^{(2)}\, K_{\text{Moyal}}\big(X^{(1)}, X^{(2)}, X^{(3)}\big)\mathcal{F}_1[X^{(1)}]\mathcal{F}_2[X^{(2)}]\,, \tag{6.4a}$$

where the measure $\mathcal{D}X$ is defined in (4.5) and the continuous Moyal kernel K_{Moyal} is of the form

$$K_{\text{Moyal}}\big(X^{(1)}, X^{(2)}, X^{(3)}\big) = \det\big[\theta(\kappa)\big]^{-2D} \exp\left[2i\int_0^\infty \frac{d\kappa}{\theta(\kappa)} \sum_{\mu=0}^{D-1} x^{(r)}_{e,\mu}(\kappa)\chi^{rs}x^{(s)}_{o,\mu}(\kappa)\right]. \tag{6.4b}$$

Here the determinant should be understood as in (5.6b). We also include in this definition a normalization factor $\det\big[\theta(\kappa)\big]^{-2D}$. Despite the fact that it is an infinite quantity this factor is needed to obtain a correct limit as $\kappa \to 0$, when we get a commutative mode. One might think that the kernels (6.4b) and (6.3b) differ due to the factor π^2, but actually this difference is only because of the difference in the measure normalization we use in (6.4a) and (6.3a).

The combined multiplication kernel that we obtained differs from (6.4b) by a linear exponent factor present in the ghost kernel (4.9a) and by an additional normalization factor $\mathcal{C}$:

$$\mathcal{C} = \frac{\mathcal{K}^D_{Q=0}\mathcal{K}_Q}{\det\big[\theta(\kappa)\big]^{-2(D+1)}} = \det\left[2\cdot\frac{12+\theta^2(\kappa)}{16}\right]^{D+1} \exp\left[-\frac{\epsilon Q^2}{24}\sum_{n=1}^{\infty} J_n^2 + \frac{\epsilon Q^2}{3}\log\gamma\right]. \tag{6.5}$$

Our results can therefore by summarized in the following mapping establishing an isomorphism of algebras

$$\mathcal{I} : \mathcal{C}^{-1}\cdot\exp\left[-i\sqrt{2}\epsilon\Big(q+\frac{Q}{2}\Big)\int_0^\infty d\kappa\, J_\kappa x_e(\kappa)\right]\Psi_q[X] \quad \mapsto \quad \mathcal{F}_q[X]\,, \tag{6.6}$$

that is $\mathcal{I}$ specifies a field redefinition that maps the Witten star product into the canonically normalized continuous Moyal product (6.4b). In view of the investigations made in the previous sections the redefinition involves an infinite multiplicative factor (same problem was noted in the matter sector in [5]). We could in principle remove this factor from the kernel and hide it into the $\mathcal{D}X$ functional integration measure, but then it would show up again in front of normalized wave functionals, like the one representing the vacuum state. We are not sure though at the current stage of investigation whether this infinite factor is a serious drawback of the continuous Moyal formalism in SFT. One should be able to do computations keeping these factors regulated and finite.

7. Conclusions

Here we summarize the main results of the paper.

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We diagonalized the 3-string vertex for a general bosonized Bose/Fermi ghost system (see (3.9)). Our results can be easily generalized for the non-zero momentum matter sector (see appendix D). In the diagonal basis the string star product was rewritten in a mixed coordinate/momentum representation and expressions for the kernel and its normalization constant (see (4.9)) were obtained. We showed that this kernel defines an associative multiplication. In particular our computations reveal that the correct midpoint insertion operator is necessary for the associativity.

One of the side issues considered along the way was a non-perturbative proof of the statement that the bpz inner product of any two surface states (with appropriate ghost insertions) is equal to one in CFT with vanishing central charge. We considered in detail the computation of an overlap of the wedge state $|0\rangle \star |0\rangle$ with itself. It was shown that for the critical bosonic string this inner product is finite. We also discussed calculation of finite parts in the identity.

The main issue considered in the paper is an isomorphism between the Witten star product algebra and a canonically normalized continuous Moyal algebra. An explicit expression for the constant relating these to algebras was found (6.5). Modulo some potential subtleties having to do with the finite part of spectral density this constant appears to be divergent.

One of the unsolved problems among others discussed in this paper is to obtain an analytic expression for the finite part of the spectral density (5.7).

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A. Relations involving vector J_n

The generating function for the vector J_n can be obtained directly from the expression (2.5e) and is of the form

$$J(z) = \sum_{n=1}^{\infty} J_n \frac{z^n}{\sqrt{n}} = -\frac{1}{\sqrt{2}} \log(1+z^2)\,. \tag{A.1}$$

Now comparing this function with the generating function [6, (B.26)] and using the expressions (3.17a) and [6, (3.17a) and (B.21)] one can obtain the following representation for the vector J

$$J = -\mathcal{P}\frac{1}{C'U'+1}W\,. \tag{A.2}$$

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Notice also that vector J is an even one

$$C'J = J\,. \tag{A.3}$$

Using the representation (A.2) one can obtain the following useful relations involving vector J

$$\begin{aligned} U'J &= -\bar{W} - J \\ \bar{U}'J &= -W - J\,. \end{aligned} \tag{A.4a}$$

We also need to know the inner product of J with W

$$\sum_{n=1}^{\infty} W_n J_n = \sum_{n=1}^{\infty} \bar{W}_n J_n = -V'_{00}\,. \tag{A.5}$$

This relation can be obtained using diagonal representation of J and W

$$\begin{aligned} \sum_{n=1}^{\infty} W_n J_n &= -\int_{-\infty}^{\infty} \frac{d\kappa}{\mathcal{N}(\kappa)}\,\frac{2}{\kappa^2}(1+\nu(\kappa)) \equiv -\int_{-\infty}^{\infty} \frac{d\kappa}{\kappa^2\mathcal{N}(\kappa)}\left[2+\nu(\kappa)+\bar{\nu}(\kappa)\right] \\ &= -2\int_0^{\infty} \frac{d\kappa}{\kappa^2\mathcal{N}(\kappa)}\left[1+3\mu(\kappa)\right] = -V'_{00}\,. \end{aligned} \tag{A.6}$$

B. Diagonalization of 3-string vertex

Let us rewrite the terms appearing in the exponential (2.5a) in the basis $e^{\dagger}_{\kappa}$, $o^{\dagger}_{\kappa}$. The quadratic part gets the following from in the diagonal basis

$$\sum_{r,s}\sum_{n,m=1}^{\infty} a_n^{(r)\dagger} V'^{rs}_{nm} a_m^{(s)\dagger} = \int_{-\infty}^{\infty} d\kappa\, C' a_{\kappa}^{(r)\dagger} \mu^{rs}(\kappa) a_{\kappa}^{(s)\dagger} = \int_0^{\infty} d\kappa\, a^{(r)\dagger}_{\kappa,\alpha} V^{rs}{}_{\alpha\beta} a^{(s)\dagger}_{\kappa,\beta}\,, \tag{B.1a}$$

where $a^{\dagger}_{\kappa,\alpha} = (e^{\dagger}_{\kappa}, o^{\dagger}_{\kappa})_{\alpha}$ and

$$\begin{aligned} V^{rs}{}_{\alpha\beta} &= \mu\delta_{\alpha\beta}\otimes\delta^{rs} + \mu_s\delta_{\alpha\beta}\otimes\varepsilon^{rs} + i\mu_a\epsilon_{\alpha\beta}\otimes\chi^{rs}\,; \\ \mu &= \mu^{11} = \frac{-1+t^2}{3+t^2}\,, \qquad \mu_s = \frac{1}{2}(\mu^{12}+\mu^{21}) = \frac{2}{3+t^2} \\ \mu_a &= \frac{1}{2}(\mu^{12}-\mu^{21}) = \frac{2t}{3+t^2}, \qquad t = \tanh\frac{\pi\kappa}{4}\,. \end{aligned} \tag{B.1b}$$

The matrices $\epsilon_{\alpha\beta}$, ε^{rs} and χ^{rs} are defined in (3.10c).

The part linear in a_n and q gets the form

$$\begin{aligned} \sum_{r,s}\sum_{n=1}^{\infty} q^{(r)} V'^{rs}_{0n} a_n^{(s)\dagger} &= -\frac{\sqrt{2}}{3}\int_{-\infty}^{\infty} \frac{d\kappa}{\kappa\sqrt{\mathcal{N}(\kappa)}} q^{(r)}\Big(3\delta^{rs} - 2 + 3\mu^{rs}\Big) a_{\kappa}^{(s)\dagger} \\ &= \int_0^{\infty} \frac{d\kappa}{\kappa\sqrt{\mathcal{N}(\kappa)}}\left[-(1+3\mu)\Big(q^{(r)}+\frac{Q}{3}\Big)e_{\kappa}^{(s)\dagger} - 2i\mu_a q^{(r)}\chi^{rs} o_{\kappa}^{(s)\dagger}\right] \\ &= \int_0^{\infty} \frac{d\kappa}{\kappa\sqrt{\mathcal{N}(\kappa)}}\, 2\mu_a\left[-t\Big(q^{(r)}+\frac{Q}{3}\Big)e_{\kappa}^{(s)\dagger} - iq^{(r)}\chi^{rs} o_{\kappa}^{(s)\dagger}\right]. \end{aligned} \tag{B.2}$$

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The term corresponding to the midpoint insertion has the following form in the diagonal basis

$$\sum_{n=1}^{\infty} J_n a_n^{(r)\dagger} = \sqrt{2}\int_{-\infty}^{\infty} \frac{d\kappa}{\kappa\sqrt{\mathcal{N}(\kappa)}}\, a_\kappa^{(r)\dagger} \overset{(3.6)}{=} 2\int_0^{\infty} \frac{d\kappa}{\kappa\sqrt{\mathcal{N}(\kappa)}}\, e_\kappa^{(r)\dagger}\,. \tag{B.3}$$

The divergent term reads

$$\sum_{n=1}^{\infty} J_n^2 = \int_{-\infty}^{\infty} \frac{d\kappa}{\mathcal{N}(\kappa)}\,(v^{(\kappa)}, J)^2 = 2\int_{-\infty}^{\infty} \frac{d\kappa}{\kappa^2\mathcal{N}(\kappa)} = 2\int_0^{\infty} d\kappa\, J_\kappa^2\,, \tag{B.4}$$

where J_κ is defined by (3.10g)

The term that depends only on the momentum can be rewritten in the integral representation using expression (A.6)

$$V_{00}'\left(\sum_r q_r^2 - Q^3\right) = \int_0^{\infty} d\kappa\, J_\kappa^2(1+3\mu)\left(\sum_r q_r^2 - Q^2\right). \tag{B.5}$$

C. Calculations of the kernel

For simplicity we will calculate the kernel K in basis (4.1) for fixed κ. This means that we will drop integration over κ in the proceeding calculations. The kernel defining multiplication in the coordinate representation has the following form (we assume $\sum_r q_r + Q = 0$)

$$\begin{aligned}
K_{\{q_r\}}(X^{(1)}, X^{(2)}, X^{(3)}) &= \left(\langle X_\kappa^{(1)}| \otimes \langle X_\kappa^{(2)}| \otimes \langle X_\kappa^{(3)}|\right)|V_3\rangle \\
&= \int \mathcal{D}a\mathcal{D}a^* \exp\left[-\frac{\epsilon}{2}\begin{pmatrix} a & a^*\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & V\end{pmatrix}\begin{pmatrix} a\\ a^*\end{pmatrix} - \right. \\
&\qquad \left. - \epsilon\begin{pmatrix} -i\sqrt{2}x & \Phi\end{pmatrix}\begin{pmatrix} a\\ a^*\end{pmatrix}\right] \times \\
&\quad \times \exp\left[-\frac{\epsilon}{2}\sum_r (x^{(r)}, x^{(r)}) - \frac{\epsilon}{4}J_\kappa^2(1+3\mu)\Big(\sum_r q_r^2 - Q^2\Big)\right] \\
&= \det{}^{-1/2}(1+V)\exp\left[\frac{\epsilon}{2}\begin{pmatrix} -i\sqrt{2}x & \Phi\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & V\end{pmatrix}^{-1}\begin{pmatrix} -i\sqrt{2}x\\ \Phi\end{pmatrix}\right] \times \\
&\quad \times \exp\left[-\frac{\epsilon}{2}\sum_r (x^{(r)}, x^{(r)}) - \frac{\epsilon}{4}J_\kappa^2(1+3\mu)\Big(\sum_r q_r^2 - Q^2\Big)\right].
\end{aligned}$$

Notice that

$$\begin{pmatrix}-1 & 1\\ 1 & V\end{pmatrix}^{-1} = (1+V)^{-1}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}$$

and

$$(1+V)^{-1} = \frac{1}{2}\otimes 1 - \frac{i}{2t}\epsilon\otimes\chi\,. \tag{C.1}$$

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Substitution yields the following expression for the kernel

$$K_{\{q_r\}}(x^{(1)},x^{(2)},x^{(3)}) = \det{}^{-1/2}(1+V)\times \\ \times\exp\left[\frac{\epsilon}{2}\sum_r (x^{(r)},x^{(r)}) - \epsilon\, x(1+V)^{-1}x - i\epsilon\sqrt{2}\,x(1+V)^{-1}\Phi + \right. \\ \left. + \frac{\epsilon}{2}\Phi(1+V)^{-1}\Phi \;-\frac{\epsilon}{4}J_\kappa^2(1+3\mu)\Big(\sum_r q_r^2 - Q^2\Big)\right]. \quad \text{(C.2)}$$

We further obtain the xx-term:

$$x(1+V)^{-1}x = \frac{1}{2}\sum_r (x^{(r)},x^{(r)}) - \frac{i}{t}\Big(x^{(1)}\wedge x^{(2)} + x^{(2)}\wedge x^{(3)} + x^{(3)}\wedge x^{(1)}\Big), \quad \text{(C.3a)}$$

the $x\Phi$-term:

$$\sqrt{2}\,x(1+V)^{-1}\Phi \;=\; \frac{1}{\sqrt{2}}\sum_r\Big(x_e^{(r)}\Phi_e^{(r)} + x_o^{(r)}\Phi_o^{(r)}\Big) - \frac{1}{\sqrt{2}}\frac{i}{t}\Big(\Phi_e^{(r)}\chi^{rs}x_o^{(s)} - \Phi_o^{(r)}\chi^{rs}x_e^{(s)}\Big) \\ \overset{(3.10f)}{=} \frac{1}{\sqrt{2}}\left[x_e^{(r)}\Phi_e^{(r)} + \frac{i}{t}\Phi_o^{(r)}\chi^{rs}x_e^{(s)}\right] = \sqrt{2}J_\kappa\sum_r x_e^{(r)}\Big(q^{(r)}+\frac{Q}{2}\Big), \quad \text{(C.3b)}$$

the $\Phi\Phi$-term:

$$\Phi(1+V)^{-1}\Phi \;=\; \frac{1}{2}\Big(\Phi_e^{(r)}\Phi_e^{(r)} + \Phi_o^{(r)}\Phi_o^{(r)}\Big) - \frac{i}{t}\Phi_e^{(r)}\chi^{rs}\Phi_o^{(s)} \\ \overset{(3.10f)}{=} \frac{1}{2}\Big(\Phi_e^{(r)}\Phi_e^{(r)} - \Phi_o^{(r)}\Phi_o^{(r)}\Big) = \frac{1}{2}J_\kappa^2(1+3\mu)\left[\sum_r q_r^2 - \frac{Q^2}{3}\right] + \frac{Q^2}{6}J_\kappa^2$$

while the determinant reads

$$\det(1+V)^{-1} = \det\left(\frac{1}{2}\otimes 1 - \frac{i}{2t}\epsilon\otimes\chi\right) = \frac{1}{2^6}\det\left[1\otimes 1 - \frac{i}{t}\epsilon\otimes\chi\right]. \quad \text{(C.3c)}$$

We used the following trick to calculate the determinant:

$$\log\det\left[1\otimes 1 - \frac{i}{t}\epsilon\otimes\chi\right] = \operatorname{Tr}\log\left[1\otimes 1 - \frac{i}{t}\epsilon\otimes\chi\right] = -\operatorname{Tr}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{(-i)^n}{t^n}\epsilon^n\otimes\chi^n \\ = -\sum_{n=1}^{\infty}\frac{1}{n}\,t^{-2n}\operatorname{Tr}\chi^{2n} = -\sum_{n=1}^{\infty}\frac{1}{n}\,t^{-2n}(-3)^{n-1}(-2\cdot 3) \\ = 2\log\left[1+\frac{3}{t^2}\right]. \quad \text{(C.3d)}$$

To obtain this we use $\chi^{2n} = (-3)^{n-1}(\varepsilon - 2)$, $\epsilon^{2n} = (-1)^n\mathbf{1}_2$. Combining equations (C.3c) and (C.3d) one obtains the following formula for the determinant

$$\det(1+V) = 4\left(\frac{4t^2}{t^2+3}\right)^2 \equiv 4(1+3\mu)^2. \quad \text{(C.3e)}$$

Substitution of (C.3) into (C.2) yields

$$K_{\{q_r\}}(x^{(r)}) = \frac{1}{2(1+3\mu)}\exp\left[\frac{i\epsilon}{t}\Big(x^{(1)}\wedge x^{(2)} + x^{(2)}\wedge x^{(3)} + x^{(3)}\wedge x^{(1)}\Big) - \right. \\ \left. - 2i\epsilon\lambda_\kappa\sum_r x_e^{(r)}\Big(q^{(r)}+\frac{Q}{2}\Big) + \left(\frac{\epsilon Q^2}{24}\sum_{n=1}^{\infty}J_n^2 - \frac{\epsilon Q^2}{3}\log\gamma\right)\right].$$

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D. Non-zero momentum matter 3-string vertex

The aim of this appendix is to adapt the formulae obtained in sections 3 and 4 to the non-zero momentum 3-string matter vertex. Essentially all we need to do is to put $Q = 0$, substitute $p_r = -q_r$, change the Kronecker symbol to the Dirac delta function, change all sums to the integrals and substitute $\epsilon \mapsto g_{\mu\nu}$.

The matter 3-string vertex in the diagonal basis has the following form

$$
\begin{aligned}
|V_3^{(\mathrm{matter})}\rangle_{123} \;=\; & \frac{1}{(2\pi)^{D/2}} \int d^D p^{(1)} d^D p^{(2)} d^D p^{(3)} \, (\deg g)^{-1} \delta(p^{(1)} + p^{(2)} + p^{(3)}) \times \\
& \times \exp\left[\int_0^\infty d\kappa \left(-\frac{1}{4} J_\kappa^2 (1+3\mu) p_\mu^{(r)} g^{\mu\nu} p_\nu^{(r)} - \right.\right. \\
& \left.\left. - \sum_r \Phi_{\alpha,\mu}^{(r)}(\kappa) g^{\mu\nu} a_{\kappa,\alpha,\mu}^{(r)\dagger} - \frac{1}{2} g^{\mu\nu} a_{\kappa,\alpha,\mu}^{(r)\dagger} V_{\kappa,\alpha\beta}^{rs} a_{\kappa,\beta,\nu}^{(s)\dagger} \right) \right] \bigotimes_{r=1}^{3} |p^{(r)}\rangle \,, \qquad \text{(D.1)}
\end{aligned}
$$

where $V_{\kappa,\alpha\beta}^{rs}$ is the same as for the ghost part and

$$
\begin{aligned}
\Phi_{e,\mu}^{(r)}(\kappa) \;&=\; -\frac{1}{2} J_\kappa (1+3\mu) p_\mu^{(r)} \,, \\
\Phi_{o,\mu}^{(r)}(\kappa) \;&=\; -i J_\kappa \mu_a p_\mu^{(s)} \chi^{sr} \equiv \frac{2i}{\theta(\kappa)} \, \Phi_e^{(s)} \chi^{sr} \,. \qquad \text{(D.2)}
\end{aligned}
$$

The multiplication kernel in the mixed coordinate/momentum basis has the form

$$
\begin{aligned}
K_{\{p_1,p_2\}}(X^{(r)}) \;&\equiv\; \Big(\langle X^{(1)}, p_1| \otimes \langle X^{(2)}, p_2| \otimes \langle X^{(3)}, -p_1 - p_2| \Big) |V_3\rangle \\
&=\; \mathcal{K}_{Q=0}^D \exp\left[2i \int_0^\infty \frac{d\kappa}{\theta(\kappa)} \, g^{\mu\nu} x_{e,\mu}^{(r)}(\kappa) \chi^{rs} x_{o,\nu}^{(s)}(\kappa) \right] \times \\
&\quad \times \exp\left[i\sqrt{2} \int_0^\infty d\kappa \, J_\kappa \left\{ x_{e,\mu}^{(1)}(\kappa) p_1^\mu + x_{e,\mu}^{(2)}(\kappa) p_2^\mu - x_{e,\mu}^{(3)}(\kappa) \bar{p}_3^\mu \right\} \right] , \qquad \text{(D.3)}
\end{aligned}
$$

where $\bar{p}_3 = p_1 + p_2$.

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